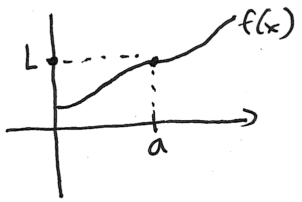
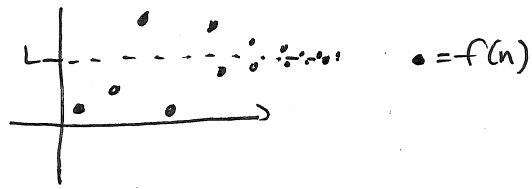
Worksheet 7 Solutions

1) a) $f(x)$ approaches L as x gets close to a vs $f(n)$ approaches L as n gets arbitrarily large.



vs



b) The terms become close to a number L as we continue the sequence.

c) • By the terms diverging to ∞ e.g. $a_n = n$ or $a_n = 2^n$

• By the terms oscillating e.g. $a_n = (-1)^n$, $a_n = \sin(n)$

d) • $a_n = \frac{1}{n^2} \to 0$ • $a_n = \dfrac{7n^2+5}{15n^3+4n^2+2} \to 0$ • $a_n = \dfrac{4n+3}{7n+5} \to \frac{4}{7}$ • $a_n = 5 + \frac{1}{n} \to 5$

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2) a) $n = 0,1,2,3,4$: $1, \frac{1}{2}, \frac{1}{2}, \frac{6}{8} = \frac{3}{4}, \frac{24}{16} = \frac{3}{2}, \ldots$

b) $n = 0,1,2,3,4$: $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$

c) $n = 1,2,3,4,5$: $1, -1, 1, -1, 1$

d) $n = 1,2,3,4,5$: $3, \frac{3}{2}, 1, \frac{3}{4}, \frac{3}{5}$

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3) a) One solution: $\{a_n\}_{n=1}^{\infty}$ $a_n = \dfrac{(-1)^{n-1}}{n^3}$

b) One solution: $\{a_n\}_{n=1}^{\infty}$ $a_n = \frac{1}{2^n} \cdot 2$

c) One solution : $\{a_n\}_{n=0}^{\infty}$ $\dfrac{(-1)^n + 1}{2} = a_n$

4) No, see $a_n = \sin(n)$ or $a_n = (-1)^n$.

5) a) Yes: $\displaystyle \lim_{n\to\infty} \frac{a_n \cdot c_n}{b_n + 1} = \frac{\lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} c_n}{\lim_{n\to\infty} b_n + 1} = \frac{15 \cdot 1}{0+1} = \boxed{15}$

b) Yes: $\displaystyle \lim_{n\to\infty} \frac{a_n + 3c_n}{2b_n + 2} = \frac{\lim_{n\to\infty} a_n + 3 \lim_{n\to\infty} c_n}{2 \lim_{n\to\infty} b_n + 2} = \frac{15+3}{0+2} = \boxed{9}$

6) a) $a_0 = 0 \qquad a_{n+1} = 3a_{n-1} + a_n^2$

$a_1 = 1$

$a_2 = 3a_0 + a_1^2 = 0 + 1 = 1$

$a_3 = 3a_1 + a_2^2 = 3 + 1 = 4$

$a_4 = 3a_2 + a_3^2 = 3 + 16 = 19$

b) $a_1 = 6 \qquad a_{n+1} = \frac{a_n}{n}$

$a_2 = \frac{a_1}{1} = \frac{6}{1} = 6$

$a_3 = \frac{a_2}{2} = \frac{6}{2} = 3$

$a_4 = \frac{a_3}{3} = \frac{3}{3} = 1$

$a_5 = \frac{a_4}{4} = \frac{1}{4}$

c) $a_1 = 2 \qquad a_{n+1} = \frac{a_n}{a_n + 1}$

$a_2 = \frac{a_1}{a_1 + 1} = \frac{2}{2+1} = \frac{2}{3}$

$a_3 = \frac{a_2}{a_2 + 1} = \frac{2/3}{\frac{2}{3} + 1} = \frac{\frac{2}{3}}{\frac{5}{3}} = \frac{2}{5}$

$a_4 = \frac{a_3}{a_3 + 1} = \frac{2/5}{7/5} = \frac{2}{7}$

$a_5 = \frac{a_4}{a_4 + 1} = \frac{2/7}{9/7} = \frac{2}{9}$

Can you find a formula for a_n?

d) $a_1 = 2 \qquad a_{n+1} = a_n - a_{n-1}$

$a_2 = 1$

$a_3 = a_2 - a_1 = 1 - 2 = -1$

$a_4 = a_3 - a_2 = -1 - 1 = -2$

$a_5 = a_4 - a_3 = -2 - (-1) = -1$

$a_6 = a_5 - a_4 = -1 - (-2) = 1$

$a_7 = a_6 - a_5 = 1 - (-1) = 2$

$a_8 = a_7 - a_6 = 2 - 1 = 1$

What is a_9? a_{11}? a_{17}?

7) Solution 1: Compute some terms.

$a_1 = 1$

$a_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right)$

$= \frac{3}{2} = 1.5$

$a_3 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{3/2}\right)$

$= \frac{17}{12} = 1.4167$

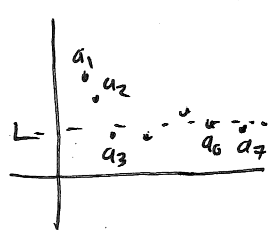
$a_4 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{17/12}\right)$

$= \frac{577}{408} = 1.414216$

$\approx \sqrt{2}$

So we guess the sequence converges to $\boxed{\sqrt{2}}$

Solution 2: Suppose $\lim_{n\to\infty} a_n = L$



Well, renaming the points doesn't change the limit, so also $\lim_{n\to\infty} a_{n-1} = L$.

But from the recursion and limit laws:

$$L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{2}\left(a_{n-1} + \frac{2}{a_{n-1}}\right)$$

$$L = \frac{1}{2}\left(L + \frac{2}{L}\right)$$

$$L^2 = \frac{1}{2}L^2 + 1$$

$$\frac{1}{2}L^2 = 1$$

$$L^2 = 2$$

$$\boxed{L = \sqrt{2}}$$